Linear interpolation of normals and norms

Supplementary materials to “Anisotropic geodesics for live-wire mesh segmentation”

We give details on linear interpolation of tensor-based anisotropic norms over a triangular mesh. While the edge refinement in mesh embedding (Section 4.2) only requires interpolation along a triangle edge, we nevertheless give a general scheme for any point along an edge or on a triangle using barycentric interpolation.

Since interpolation is piece-wise linear, we only need to consider one triangle. As input, each vertex of the triangle p_i ($i = 1, 2, 3$) is associated with a unit outward normal n_i and a pair of orthogonal vectors u_i, w_i on the tangent plane that together define a right-handed coordinate system. The norm at each vertex is defined by a tensor M_i, whose j-th eigenvalue ($j = 1, 2$) and eigenvector (a 2D vector in the coordinate system defined by u_i, w_i) is denoted respectively as $\lambda_{i,j}$ and $e_{i,j}$.

Given a point p inside the triangle, we express it as a linear combination of the three vertices

$$p = \sum_{i=1}^{3} w_i p_i$$

where w_i are the barycentric coordinates. We first get the normal vector n at p by interpolating the vertex normals (as done in Phong shading),

$$n = \sum_{i=1}^{3} w_i n_i / \| \sum_{i=1}^{3} w_i n_i \|$$

To interpolate the tensors M_i, which are defined on different tangent planes, we first need to move them to the tangent plane at p. We utilize *parallel transport*. Given a pair of vectors b, b', parallel transport computes a minimal twist rotation that aligns the tangent planes of the two vectors. Given a vector a on the tangent plane of b, the result of parallel transport of a to the tangent plane of b' can be computed as

$$trans(a, b, b') = a - \frac{2((b + b') \cdot a)(b + b')}{(b + b')^2}$$

We first transport the tangent frame axes u_i, w_i at each vertex to the tangent plane at p, as

$$u'_i = trans(u_i, n_i, n), \quad w'_i = trans(w_i, n_i, n).$$

Using some chosen but fixed pair of axes u, w on the tangent plane at p, we can express each u'_i and w'_i as a linear combination of u, w, thereby rewriting the 2D eigenvectors $e_{i,j}$ in the coordinate axes of u, w. We will denote these rewritten vectors as $e'_{i,j}$. We can then reconstruct the transported tensors M'_i on the tangent plane of p as

$$M'_i = \lambda_{i,1} < e'_{i,1}, e'_{i,1} > + \lambda_{i,2} < e'_{i,2}, e'_{i,2} >$$

where $< \cdot, \cdot >$ denotes the outer product. Finally, we interpolate the three transported tensors to obtain the tensor M at p,

$$M = \sum_{i=1}^{3} w_i M'_i.$$